INTEGRATION IN A CONVEX LINEAR TOPOLOGICAL SPACE

BY

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Introduction

The results of this paper center around the definition of an integration process for multi-valued set functions which are defined over a σ-field $\mathfrak{M}$ and whose values lie in a convex linear topological space $\mathfrak{X}$. As such they represent a substantial generalization of the basic results contained in a paper by A. Kolmogoroff [7](1), who considered the case in which $\mathfrak{X}$ is the real numbers. On the other hand, the method of defining the integral is a generalization of that used by R. S. Phillips(2) [12, p. 118], although Phillips considered integration only with respect to a positive numerical measure function and restricted the integral to be single-valued. The importance of the Phillips definition lies in the fact that it relieves one of the necessity of considering infinite sums. Throughout the paper is emphasized a type of convergence for sets in a linear topological space which is analogous to the Hausdorff notion of convergence for sets in a metric space [5, §28]. G. B. Price has made a similar use of the Hausdorff convergence for sets [13, Parts II, V].

The contents of the paper are divided into four parts. Part I (§§1–3) contains a short discussion of convex linear topological spaces, a definition of the notion of unconditional summability, which plays a central role in the definition of the integral, and two theorems on additive set functions. Part II (§§4–9) contains the general theory of the $\mathcal{V}$- and $\mathcal{S}\mathcal{V}$-integrals. The $\mathcal{V}$-integral is multi-valued and is defined for multi-valued set functions $F(\sigma)$. The $\mathcal{S}\mathcal{V}$-integral is the single-valued specialization of the $\mathcal{V}$-integral. Definitions and basic properties of these integrals account for §§4, 5. Section 6 contains a discussion of a generalization of the Kolmogoroff [7] notion of differential equivalence applied to the $\mathcal{V}$-integral, and §7 contains a proof that the transform of an integrable function by a general type of linear transformation is integrable. In §8 it is shown that the definition of the $\mathcal{S}\mathcal{V}$-integral can be weakened in case $\mathfrak{X}$ is complete in a certain sense. Section 9 contains a convergence theorem for the $\mathcal{S}\mathcal{V}$-integral which involves a generalization of the notion of approximate convergence to functions $F(\sigma)$ of the type considered here. The approximate convergence is relative to a positive numerical measure function $m(\sigma)$ whose only relation to the integral lies in the condition that

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) See also a paper by Garrett Birkhoff [2, p. 51] where the same definition used by Phillips is given.

the integrals of the functions considered be absolutely continuous relative to $m(\sigma)$. Part III (§§10–13) is concerned with the $\mathcal{V}_B$-integral which is a specialization of the $\mathfrak{S}\mathcal{V}$-integral and may be described as integration with respect to a "bilinear" function. The "bilinear" function $B[y, \sigma]$ is a generalization of $m(\sigma)y$, where $m(\sigma)$ is a numerical measure function. It has its values in $\mathfrak{X}$ and is defined for y in a linear space $\mathfrak{Y}$ and σ in the σ-field $\mathfrak{M}$. It is linear in y for each σ and completely additive in σ for each y. Functions $y(\sigma)$ to be integrated have their values in $\mathfrak{Y}$ while the value of the integral is in $\mathfrak{X}$. Section 10 contains a definition of the $\mathcal{V}_B$-integral and a discussion of its fundamental properties. In §11 the $\mathcal{V}_B$-integral is considered for the case in which $\mathfrak{X}$ is complete in the sense of §8. Section 12 contains a discussion of absolute continuity and a convergence theorem for the $\mathcal{V}_B$-integral. In §13 the existence of the $\mathcal{V}_B$-integral is proved for a certain class of measurable functions. Part IV (§§14–16) relates the above integrals to previously defined integrals. For the case in which $\mathfrak{X}$ is the real numbers, the $\mathfrak{S}\mathcal{V}$-integral includes an integral of Kolmogoroff [7]. The $\mathcal{V}_B$-integral reduces in a special case to an integral of Phillips [12], and a specialization of the $\mathcal{V}_B$-integral includes an integral of Price [13]. Relation of the $\mathcal{V}_B$-integral to the various other integrals which have been defined can be obtained through its relation to the Phillips integral (see [12, §7]).

Part I. Preliminary considerations

1. **Convex linear topological spaces.** The type of linear topological space $\mathfrak{X}$ to be considered here is that introduced by J. von Neumann [11, p. 4]. It is defined as follows:

A set $\mathfrak{X}$ of elements x is said to constitute a *linear topological space* provided it is linear(3) (Banach [1, p. 26]) and provided it contains a family $\mathcal{V}$ of subsets such that

(1) $\theta \in V$ for every $V \in \mathcal{V}$.

(2) $x \in V$ for every $V \in \mathcal{V}$ implies $x = \theta$.

(3) $V_1, V_2 \in \mathcal{V}$ implies the existence of $V_3 \in \mathcal{V}$ such that(4) $V_3 \subset V_1 \cap V_2$.

(4) $V \in \mathcal{V}$ implies the existence of $V' \in \mathcal{V}$ such that(5) $V' + V' \subset V$.

(5) $V \in \mathcal{V}$ implies the existence of $V' \in \mathcal{V}$ such that(I) $\alpha V' \subset V$ for all $|\alpha| \leqq 1$.

(6) $x \in \mathfrak{X}$ and $V \in \mathcal{V}$ imply the existence of α such that $x \in \alpha V$.

Also, $\mathfrak{X}$ is said to be *convex* provided

(7) $V \in \mathcal{V}$ implies $V + V \subset 2V$.

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(3) Scalar multipliers are assumed real. The zero element will be denoted by θ.

(4) The symbols $\subset$, $\cap$, $\cup$ denote, respectively, set-theoretic "included in," "intersection" and "union," and will be used with their usual variations throughout the paper. $A \cap CB$ denotes the set of points contained in A but not in B.

(5) $V_1 + V_2 = \{x_1 + x_2 \mid x_1 \in V_1, x_2 \in V_2\}$, where $\{x \mid P\}$ denotes the set of all elements x subject to the condition P. Similarly, $\alpha V = \{\alpha x \mid x \in V\}$.

A set G in $\mathfrak{X}$ is defined to be *open* provided, for every $x \in G$, there exists $V \in \mathcal{V}$ such that $x + V \subset G$. The *interior* of a set X is defined by $X_i = \{x \mid x + V \subset X \text{ for some } V \in \mathcal{V}\}$. X_i is evidently open. A set is defined to be *closed* if it is the complement of an open set, and the *closure* of a set X is defined by $X_{cl} = C((CX)_i)$. Evidently X_{cl} is closed. The above class of open sets defines a regular Hausdorff topology in $\mathfrak{X}$ so that the operations of addition and multiplication by a number are continuous [11, Theorem 6]. It is known (Wehausen [16, Theorem 1]) that this topology is equivalent to that introduced by Kolmogoroff [8, p. 29]. In all that follows $\mathfrak{X}$ will be assumed to be a convex linear topological space as above defined. An important consequence of the convexity of the space $\mathfrak{X}$ is that the closure V_{cl} of every $V \in \mathcal{V}$ is a convex set; that is, $C_o V_{cl} = V_{cl}$, where $C_o X = \{\sum_{i=1}^n \alpha_i x_i \mid x_i \in X,\ \alpha_i \geqq 0,\ \sum_{i=1}^n \alpha_i = 1,\ n \text{ arbitrary}\}$. This implies that $\sum_{i=1}^n \alpha_i \mathcal{V}_{cl} = (\sum_{i=1}^n \alpha_i) V_{cl}$ for arbitrary $\alpha_i \geqq 0$. Another important consequence of the convexity of $\mathfrak{X}$ is that $0 < \alpha < \beta$ implies $\alpha V_{cl} \subset \beta V$ for every $V \in \mathcal{V}$. In much of the material which follows, the assumption that $\mathfrak{X}$ is convex could be avoided; however computation is greatly simplified by using it and some of the important theorems (for example, Theorems 3.1, 5.5, 9.5) seem to involve it rather deeply.

It will be desirable later(6) to subject $\mathfrak{X}$ to a completeness condition which we introduce here. It is convenient to define the condition in terms of the Moore-Smith [10, p. 103] general limits notion on a class $\mathcal{L}$ with a transitive compositive relation R on $\mathcal{L}\mathcal{L}$. A set $\{x_l\}$ of elements in $\mathfrak{X}$, where l ranges over $\mathcal{L}$, is called an *$\mathcal{L}$-directed* set. $\{x_l\}$ is called a *fundamental* $\mathcal{L}$-directed set provided, for every $V \in \mathcal{V}$, there exists l_V such that $l_i R l_V$ $(i = 1, 2)$ implies $x_{l_1} - x_{l_2} \in V$. The space $\mathfrak{X}$ is said to be *complete relative to* $\mathcal{L}$ provided every fundamental $\mathcal{L}$-directed set converges (in the Moore-Smith sense) to an element of $\mathfrak{X}$. We will be interested in two important specializations of this completeness notion, for example, the case where $\mathcal{L}$ is the set of positive integers and R is the usual order relation "$>$," which gives the ordinary sequential completeness, and the case where $\mathcal{L}$ is the family of neighborhoods $\mathcal{V}$ and R is the set-theoretic "included in" (that is, $V_1 R V_2$ means $V_1 \subset V_2$)(7). It is easy to prove that, if $\mathfrak{X}$ satisfies the first countability axiom (Hausdorff [5, p. 229]) and is sequentially complete, then it is complete relative to $\mathcal{V}$. It follows that, if $\mathfrak{X}$ is a Banach space with its norm topology (that is, $\mathcal{V}$ is the family of spheres with center θ), then $\mathfrak{X}$ is complete relative to $\mathcal{V}$.

2. **Unconditional summability.** In the following, π will always denote a finite set of positive integers and $\pi_1 \geqq \pi_2$ will mean that π_1 contains π_2. Also, $\sum_\pi$ will mean summation over those $n \in \pi$. Two subsets X, Y of $\mathfrak{X}$ are said to be *equal within* V provided $X \subset Y + V$ and $Y \subset X + V$.

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(6) See Theorems 8.5, 12.3 below.

(7) The space $\mathfrak{X}$ may be said to be *complete* provided it is complete relative to *every* $\mathcal{L}$. This condition can be shown to be equivalent to simply completeness relative to $\mathcal{V}$ (see Graves [4, p. 62]).

2.1. DEFINITION. *Two sequences* $\{X_n\}$, $\{X_n'\}$ *of subsets of* $\mathfrak{X}$ *are said to be summably equal within* V *provided there exists a* π_0 *such that* $\pi \geqq \pi_0$ *implies* $\sum_\pi X_n, \sum_\pi X_n'$ *are equal within* V.

2.2. DEFINITION. *A sequence* $\{X_n\}$ *of subsets of* $\mathfrak{X}$ *is said to be unconditionally summable (we write u.s.) to a set* X *with respect to* V *provided there exists* π_0 *such that* $\pi \geqq \pi_0$ *implies* $X, \sum_\pi X_n$ *are equal within* V.

Observe[8] that $\{X_n\}$ is u.s. to a single element $x \in \mathfrak{X}$ with respect to V provided there exists π_0 such that $\pi \geqq \pi_0$ implies $\pm \{\sum_\pi X_n - x\} \subset V$. This special case of the above notion of unconditional summability was used by R. S. Phillips [12, p. 118]. Observe also that, if each of the sets X_n, X consists of only a single element, then u.s. of $\{X_n\}$ to X with respect to every $V \in \mathcal{V}$ reduces to the ordinary unconditional convergence.

2.3. THEOREM. *In order that* $\{X_n\}$ *be u.s. to* X *with respect to* V *it is both necessary and sufficient that, for every rearrangement* $n(k)$ *of the sets* X_n, *there exists* k_0 *for which* $k \geqq k_0$ *implies* $X, \sum_{i=1}^k X_{n(i)}$ *are equal within* V.

The method of proof used in an analogous situation involving unconditional convergence of series in normed vector spaces can be applied with slight modification here (see Hildebrandt [6, p. 90]).

3. **Additive set functions.** Let M be an abstract set of elements and $\mathfrak{M}$ a σ-field[9] of subsets of M. Elements of $\mathfrak{M}$ will be denoted by σ. Also let $m(\sigma)$ be a positive, completely additive measure function defined over $\mathfrak{M}$. A single-valued function $x(\sigma)$ on $\mathfrak{M}$ to the space $\mathfrak{X}$ is said to be *completely additive* if for every sequence $\{\sigma_n\}$ of disjoint elements of $\mathfrak{M}$, the series $\sum x(\sigma_n)$ is unconditionally convergent to $x(\bigcup \sigma_n)$. $x(\sigma)$ is said to be *absolutely continuous relative to* $m(\sigma)$ provided, for every $V \in \mathcal{V}$, there exists $\delta_V > 0$ such that $x(\sigma) \in V$ whenever $m(\sigma) < \delta_V$.

3.1. THEOREM[10]. *If* $x(\sigma)$ *is completely additive on* $\mathfrak{M}$ *to* $\mathfrak{X}$ *and if* $m(\sigma) = 0$ *implies* $x(\sigma) = \theta$, *then* $x(\sigma)$ *is absolutely continuous relative to* $m(\sigma)$.

Suppose[11] the theorem not true; then there exists a $V \in \mathcal{V}$ and a sequence of elements $\sigma_k \in \mathfrak{M}$ such that $\lim_{n \to \infty} m(\bigcup_{k=n}^\infty \sigma_k) = 0$ and such that[12] $\|x(\sigma_k)\|_V > 2$ for each k. Now, by a result of Wehausen [16, Theorem 8], there exists a linear continuous operation $\bar{x}_1$ on $\mathfrak{X}$ to the real numbers such that $|\bar{x}_1(x)| \leqq \|x\|_V$ for all x and $\bar{x}_1(x(\sigma_1)) = \|x(\sigma_1)\|_V$. It is obvious that

(8) Unconditional summability obviously involves a convergence notion related to the well known Hausdorff convergence for sets in a metric space [5, §28].

(9) The σ-field $\mathfrak{M}$ has the following properties: (1) $\mathfrak{M}$ contains the empty set; (2) if $\sigma \in \mathfrak{M}$, then $M \cap C\sigma \in \mathfrak{M}$; (3) if $\sigma_n \in \mathfrak{M}$ $(n = 1, 2, \cdots)$, then $\bigcup_{n=1}^\infty \sigma_n \in \mathfrak{M}$.

(10) See Dunford [3, Theorem 42] and Kunisawa [9, pp. 68, 69].

(11) The method of proof used here was suggested to the writer by R. S. Phillips.

(12) $\|x\|_V$ is the von Neumann pseudo-norm; that is, $\|x\|_V = \max(\|x\|_V^+, \|-x\|_V^+)$, where $\|x\|_V^+ = \text{g.l.b. } \{\alpha \mid \alpha > 0, x \in \alpha V\}$ [11, pp. 18, 19].

$f(\sigma) = \bar{x}_1(x(\sigma))$ is a completely additive, real-valued function of σ such that $m(\sigma) = 0$ implies $f(\sigma) = 0$. Therefore, by a well known theorem (see Saks [15, Theorem 13.2, p. 31]), $f(\sigma)$ is absolutely continuous relative to $m(\sigma)$. It follows that there exists an n_1 such that $|f(\bigcup_{k=n}^{\infty}(\sigma_1 \cap \sigma_k))| < 1$ for $n \geqq n_1$. Now, if we take(13) $\sigma_1^0 = \sigma_1 \cap C[\bigcup_{k=n_1}^{\infty}(\sigma_1 \cap \sigma_k)]$, it is clear that $f(\sigma_1) = f(\sigma_1^0) + f(\bigcup_{k=n_1}^{\infty}(\sigma_1 \cap \sigma_k))$. But $f(\sigma_1) = \bar{x}_1(x(\sigma_1)) = \|x(\sigma_1^0)\|_V > 2$; therefore, $f(\sigma_1^0) = \bar{x}_1(x(\sigma_1^0)) > 1$. Since $|\bar{x}_1(x(\sigma_1^0))| \leqq \|x(\sigma_1^0)\|_V$, we have $\|x(\sigma_1^0)\|_V > 1$. Moreover, $\sigma_1^0 \cap \sigma_n = 0$ for all $n \geqq n_1$. Repeating the above procedure on the sequence $\sigma_{n_1}, \sigma_{n_1+1}, \cdots$ one obtains a $\sigma_2^0 \subset \sigma_{n_1}$ and an n_2 such that $\sigma_2^0 \cap \sigma_n = 0$ for $n \geqq n_2$ and $\|x(\sigma_2^0)\|_V > 1$. This process can be continued indefinitely to obtain a sequence $\{\sigma_n\}$ of *disjoint* elements of $\mathfrak{M}$ such that $\|x(\sigma_n^0)\|_V > 1$ for all n. But this result obviously contradicts the assumption that $x(\sigma)$ be completely additive; therefore $x(\sigma)$ is absolutely continuous relative to $m(\sigma)$.

3.2. THEOREM. *Let each of the functions $x_n(\sigma)$ be additive and absolutely continuous relative to $m(\sigma)$. Then, if $\{x_n(\sigma)\}$ is a fundamental sequence for each σ, the $x_n(\sigma)$ are equi-absolutely continuous relative to $m(\sigma)$.*

A proof indentical with that which gives Theorem 6.1 of the Phillips paper [12, p. 125] applies here; so will be omitted. The method of proof is due to Saks [14].

PART II. THE GENERAL THEORY

4. **Definitions of the $\mathcal{V}$- and $S\mathcal{V}$-integrals.** Let $\mathfrak{M}$ denote, as before, a σ-field of subsets of an abstract set M. A subdivision of M into a finite or denumerable number of disjoint elements of $\mathfrak{M}$ will be denoted by $\Delta = \{\sigma_i\}$, where $M = \bigcup \sigma_i$ and $\sigma_i \cap \sigma_j = 0$ $(i \neq j)$. Δ^1 is said to be *finer than* Δ^2 provided, for every σ_i^1, there exists a $\sigma_{n_i}^2$ such that $\sigma_i^1 \subseteq \sigma_{n_i}^2$; we write $\Delta^1 \geqq \Delta^2$. The product of two subdivisions is a subdivision defined by $\Delta^1\Delta^2 = \{\sigma_i^1 \cap \sigma_j^2\}$. Evidently the product of two subdivisions is finer than either one of the subdivisions. Let $\Delta_0 = \{\sigma_k\}$ be a given fixed subdivision and Δ^k $(k = 1, 2, \cdots)$ an arbitrary sequence of subdivisions; then the subdivision Δ which coincides with Δ^k on the set σ_k $(k = 1, 2, \cdots)$ is called the *sum* of the Δ^k over Δ_0.

The functions $F(\sigma)$ to be studied are multi-valued and are defined over(14) $\mathfrak{M}$ (that is, excluding the empty set) with values in $\mathfrak{X}$. Let $\Delta = \{\sigma_i\}$ be an arbitrary subdivision of M; we denote the sequence of sets $\{F(\sigma \cap \sigma_i)\}$ by the symbol $J(F, \sigma, \Delta)$.

4.1. DEFINITION. *The function $F(\sigma)$ is said to be $\mathcal{V}$-integrable over σ_0 provided there is a set $I(F, \sigma_0) \subset \mathfrak{X}$ such that, for every $V \in \mathcal{V}$, there exists $\Delta_{V\sigma_0}$ for which $\Delta \geqq \Delta_{V\sigma_0}$ implies $J(F, \sigma_0, \Delta)$ is u.s. to $I(F, \sigma_0)$ with respect to V. The*

(13) See Footnote 4 above.

(14) One could develop the following theory for functions defined over only a portion of $\mathfrak{M}$; however there would be a considerable loss of simplicity in the statement of definitions and theorems.

closure(15) *of the set* $I(F, \sigma_0)$ *will be called the* $\mathcal{V}$*-integral of* $F(\sigma)$ *over* σ_0, *and we write* $I(F, \sigma_0)_{cl}=\int_{\sigma_0}F(d\sigma)$. *Furthermore, if* $\int_{\sigma_0}F(d\sigma)$ *consists of a single element, then* $F(\sigma)$ *is said to be* $\mathfrak{S}\,\mathcal{V}$*-integrable over* σ_0.

4.2. THEOREM. *If* $F(\sigma)$ *is* $\mathcal{V}$*-integrable on* σ_0, *then* $\int_{\sigma_0}F(d\sigma)$ *is unique.*

Suppose $F(\sigma)$ $\mathcal{V}$-integrable to each of the sets $I_1(F, \sigma_0)$, $I_2(F, \sigma_0)$. Then it is immediate from the definition that, for every $V\in\mathcal{V}$, there exists a $\Delta_{V\sigma_0}$ such that $J(F, \sigma_0, \Delta)$ is u.s. to *both* $I_1(F, \sigma_0)$ and $I_2(F, \sigma_0)$ with respect to V for $\Delta\geqq\Delta_{V\sigma_0}$; that is, if $\Delta=\{\sigma_i\}$, then there exists π_0 such that $\pi\geqq\pi_0$ implies $\sum_\pi F(\sigma_0\cap\sigma_i)$, $I_m(F, \sigma_0)$ are equal within V $(m=1, 2)$. It follows immediately from this result that $I_1(F, \sigma_0)$, $I_2(F, \sigma_0)$ are equal within $2V$. Therefore, $I_k(F, \sigma_0)\subset I_l(F, \sigma_0)+2V$ $(k=1, 2;\ l=1, 2)$. Since V is arbitrary, $I_k(F, \sigma_0)\subseteq I_l(F, \sigma_0)_{cl}$; hence $I_1(F, \sigma_0)_{cl}=I_2(F, \sigma_0)_{cl}$.

Observe that, if $F(\sigma\cap\sigma_0)=\theta$ for every σ, then $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0 to the value θ.

5. **Properties of the integrals**(16).

5.1. THEOREM. *If* $F(\sigma)$, $G(\sigma)$ *are* $\mathcal{V}$*-integrable on* σ_0 *and* α *is any real number, then* $\alpha F(\sigma)$, $F(\sigma)+G(\sigma)$ *are* $\mathcal{V}$*-integrable on* σ_0 *and*

$$\int_{\sigma_0}\alpha F(d\sigma)=\alpha\int_{\sigma_0}F(d\sigma),\qquad \int_{\sigma_0}F(d\sigma)+G(d\sigma)=\left[\int_{\sigma_0}F(d\sigma)+\int_{\sigma_0}G(d\sigma)\right]_{cl}.$$

If $\alpha=0$, the statement for $\alpha F(\sigma)$ is obvious, and, if $\alpha\neq0$, the desired result is a consequence of the fact that, if $J(F, \sigma_0, \Delta)$ is u.s. to $I(F, \sigma_0)$ with respect to V, then $J(\alpha F, \sigma_0, \Delta)$ is u.s. to $\alpha I(F, \sigma_0)$ with respect to αV.

In the case of $F(\sigma)+G(\sigma)$, we observe that, for arbitrary $V\in\mathcal{V}$, there exists a $\Delta_{V\sigma_0}$ such that, if $\Delta\geqq\Delta_{V\sigma_0}$, then $J(F, \sigma_0, \Delta)$, $J(G, \sigma_0, \Delta)$ are, respectively, u.s. to $\int_{\sigma_0}F(d\sigma)$, $\int_{\sigma_0}G(d\sigma)$ with respect to V. From this it is immediate that $J(F+G, \sigma_0, \Delta)$ is u.s. to $\int_{\sigma_0}F(d\sigma)+\int_{\sigma_0}G(d\sigma)$ with respect to $2V$. Since V is arbitrary, the desired result follows.

5.2. COROLLARY. *If* $F(\sigma)$, $G(\sigma)$ *are* $\mathfrak{S}\,\mathcal{V}$*-integrable on* σ_0, *then*

$$\int_{\sigma_0}\alpha F(d\sigma)=\alpha\int_{\sigma_0}F(d\sigma),\qquad \int_{\sigma_0}F(d\sigma)+G(d\sigma)=\int_{\sigma_0}F(d\sigma)+\int_{\sigma_0}G(d\sigma).$$

5.3. THEOREM. *If* $F(\sigma)$ *is* $\mathcal{V}$*-integrable on both* σ_1, σ_2 *and if* $\sigma_1\cap\sigma_2=0$, *then* $F(\sigma)$ *is* $\mathcal{V}$*-integrable on* $\sigma_1\cup\sigma_2$ *and*

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(15) The closure of a set X is denoted by X_{cl}. It can be shown (see [11]) that $X_{cl}=\bigcap(X+V)$ for $V\in\mathcal{V}$. Observe that, if $F(\sigma)$ is $\mathcal{V}$-integrable to $I(F, \sigma_0)$, then it is also $\mathcal{V}$ integrable to $I(F, \sigma_0)_{cl}$.

(16) The theorems of this section will be stated for the $\mathcal{V}$-integral and are, of course, true for the $\mathfrak{S}\,\mathcal{V}$-integral. In case the results for the $\mathfrak{S}\,\mathcal{V}$-integral are stronger, we state them as corollaries.

$$\int_{\sigma_1 \cup \sigma_2} F(d\sigma) = \left[\int_{\sigma_1} F(d\sigma) + \int_{\sigma_2} F(d\sigma) \right]_{cl}.$$

5.4. COROLLARY. *If $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on both σ_1, σ_2 and if $\sigma_1 \cap \sigma_2 = 0$, then $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on $\sigma_1 \cup \sigma_2$ and*

$$\int_{\sigma_1 \cup \sigma_2} F(d\sigma) = \int_{\sigma_1} F(d\sigma) + \int_{\sigma_2} F(d\sigma).$$

5.5. THEOREM. *The $\mathcal{V}$-integral is a completely additive function of σ in the sense that, if $F(\sigma)$ is $\mathcal{V}$-integrable on each σ^k $(k=0, 1, 2, \cdots)$, where $\sigma^0 = \bigcup \sigma^k$ and $\sigma^m \cap \sigma^n = 0$ $(m \neq n;\ m,\ n \neq 0)$, then $\{\int_{\sigma^k} F(d\sigma)\}$ $(k=1, 2, \cdots)$ is u.s. to $\int_{\sigma^0} F(d\sigma)$ with respect to every $V \in \mathcal{V}$.*

There is no loss in taking $\sigma^0 = M$. Since the integral exists for each σ^k, there exists Δ_V^k such that, if $\Delta \geqq \Delta_V^k$, $J(F, \sigma^k, \Delta)$ is u.s. to $\int_{\sigma^k} F(d\sigma)$ with respect to $2^{-k-1}V$ $(k=0, 1, 2, \cdots)$. Denote by Δ_1 the sum of the Δ_V^k $(k=1, 2, \cdots)$ over the subdivision $\{\sigma^k\}$ (see §4 above) and set $\Delta_0 = \Delta_1 \Delta_V^0$. Evidently on the set σ^k the subdivision Δ_0 is finer than the subdivision Δ_V^k. Therefore $J(F, \sigma^k, \Delta_0)$ is u.s. to $\int_{\sigma^k} F(d\sigma)$ with respect to $2^{-k-1}V$. If $\Delta_0 = \{\sigma_i\}$, then there exists π_0 such that $\pi \geqq \pi_0$ implies the equality of $\sum_\pi F(\sigma_i)$, $\int_M F(d\sigma)$ within $V/2$. Set $n_V = \max \{n \mid \sigma^n \cap \bigcup_{\pi_0} \sigma_i \neq 0\}$; then, for an arbitrary (but fixed) $n \geqq n_V$, there exists a π_n such that $\pi_n \geqq \pi_0$, $i \in \pi_n$, $l > n$ imply $\sigma^l \cap \sigma_i = 0$ and

$$\sum_{i \in \pi_n} F(\sigma^k \cap \sigma_i) \subset \int_{\sigma^k} F(d\sigma) + 2^{-k-1}V, \quad \int_{\sigma^k} F(d\sigma) \subset \sum_{i \in \pi_n} F(\sigma^k \cap \sigma_i) + 2^{-k-1}V,$$

for $k=0, 1, 2, \cdots, n$. It is obvious that [17]

$$\sum_{k=1}^{n} \sum_{i \in \pi_n} F(\sigma^k \cap \sigma_i) = \sum_{\pi_n} F(\sigma_i);$$

therefore,

$$\sum_{\pi_n} F(\sigma_i) \subset \sum_{k=1}^{n} \int_{\sigma^k} F(d\sigma) + V/2, \quad \sum_{k=1}^{n} \int_{\sigma^k} F(d\sigma) \subset \sum_{\pi_n} F(\sigma_i) + V/2.$$

Now, since $\pi_n \geqq \pi_0$ we have immediately that

$$\sum_{k=1}^{n} \int_{\sigma^k} F(d\sigma) \subset \int_M F(d\sigma) + V, \quad \int_M F(d\sigma) \subset \sum_{k=1}^{n} \int_{\sigma^k} F(d\sigma) + V,$$

where $n \geqq n_V$ is arbitrary. This argument does not depend on the order in which the sets σ^k are taken; therefore the desired result follows from Theorem 2.3.

(17) It is understood that terms for which $\sigma^k \cap \sigma_i = 0$ are omitted.

5.6. COROLLARY. *The* $\mathfrak{S}\mathcal{V}$*-integral is completely additive in the ordinary sense.*

The proofs of the next two theorems are not difficult and will be omitted.

5.7. THEOREM. *Any function* $F(\sigma)$ *defined over* $\mathfrak{M}$ *which is completely additive in the sense of Theorem* 5.5 *is* $\mathcal{V}$*-integrable on every* σ *and* $F(\sigma)_{cl}=\int_\sigma F(d\sigma)$.

5.8. THEOREM. *If* $F(\sigma)$, $G(\sigma)$ *are* $\mathcal{V}$*-integrable on* σ_0 *and if there exists* Δ_0 *such that* $\{\sigma_i\}=\Delta\geqq\Delta_0$ *implies the existence of* π_Δ *such that, for* $\pi\geqq\pi_\Delta$, *it is true that*

$$\sum_\pi G(\sigma_0\cap\sigma_i)\subset\sum_\pi F(\sigma_0\cap\sigma_i)+V,$$

then

$$\int_{\sigma_0}G(d\sigma)\subseteq\left[\int_{\sigma_0}F(d\sigma)+V\right]_{cl}.$$

5.9. COROLLARY. *If* $F(\sigma)$, $G(\sigma)$ *satisfy the conditions of Theorem* 5.8 *and if in addition* $F(\sigma)$ *is* $\mathfrak{S}\mathcal{V}$*-integrable, then* $\int_{\sigma_0}G(d\sigma)-\int_{\sigma_0}F(d\sigma)\subseteq V_{cl}$.

5.10. THEOREM. *If* $F(\sigma)$ *is* $\mathcal{V}$*-integrable on* σ_0, *then so also is* $F(\sigma)_{cl}$ *and to the same value.*

For every $V\in\mathcal{V}$, there exists Δ_V such that, if $\{\sigma_i\}=\Delta\geqq\Delta_V$, then there exists π_Δ for which $\pi\geqq\pi_\Delta$ implies the equality of $\int_{\sigma_0}F(d\sigma)$, $\sum_\pi F(\sigma_0\cap\sigma_i)$ within V. Since $F(\sigma)_{cl}\subset F(\sigma)+V'$ for *arbitrary* $V'\in\mathcal{V}$ and σ, it follows that $\sum_\pi F(\sigma_0\cap\sigma_i)_{cl}\subset\sum_\pi F(\sigma_0\cap\sigma_i)+V$. Therefore $\int_{\sigma_0}F(d\sigma)$, $\sum_\pi F(\sigma_0\cap\sigma_i)_{cl}$ are equal within $2V$. Since V is arbitrary, the desired result follows by definition.

5.11. THEOREM. *If* $F(\sigma)$, $G(\sigma)$ *are* $\mathcal{V}$*-integrable on* σ_0 *and if, for every* $\sigma\subseteq\sigma_0$, $G(\sigma)\subseteq F(\sigma)_{cl}$, *then* $\int_{\sigma_0}G(d\sigma)\subseteq\int_{\sigma_0}F(d\sigma)$.

In the present case the conditions of Theorem 5.8 hold for every $V\in\mathcal{V}$; therefore

$$\int_{\sigma_0}G(d\sigma)\subseteq\left[\int_{\sigma_0}F(d\sigma)+V/2\right]_{cl}\subset\int_{\sigma_0}F(d\sigma)+V$$

for every $V\in\mathcal{V}$. Since $\int_{\sigma_0}F(d\sigma)$ is closed, $\int_{\sigma_0}G(d\sigma)\subseteq\int_{\sigma_0}F(d\sigma)$.

5.12. COROLLARY. *If* $F(\sigma)$ *is* $\mathfrak{S}\mathcal{V}$*-integrable on* σ_0 *and if, for* $\sigma\subseteq\sigma_0$, $G(\sigma)\subseteq F(\sigma)_{cl}$, *then* $G(\sigma)$ *is* $\mathfrak{S}\mathcal{V}$*-integrable on* σ_0 *and* $\int_{\sigma_0}G(d\sigma)=\int_{\sigma_0}F(d\sigma)$.

6. **Differential equivalence.** The results of this section parallel similar results obtained by Kolmogoroff for the case of $\mathfrak{X}$ the real numbers. The following definition of differential equivalence is a direct generalization of the Kolmogoroff definition [7, p. 666].

6.1. DEFINITION. *The functions* $F(\sigma)$, $G(\sigma)$ *are said to be differentially equivalent* (*we write d.e.*) *on the set* σ_0 *provided for every* $V\in\mathcal{V}$, *there exists a*

Δ_V such that, if $\Delta \geqq \Delta_V$, then $J(F, \sigma_0, \Delta)$, $J(G, \sigma_0, \Delta)$ are summably equal within V. (See Definition 2.1 above.)

6.2. THEOREM. *If $F(\sigma)$, $G(\sigma)$ are d.e. on σ_0, then the $\mathcal{V}$-integrability of either function on σ_0 implies the $\mathcal{V}$-integrability of the other and to the same value.*

Suppose $F(\sigma)$ $\mathcal{V}$-integrable on σ_0. Evidently, for every $V \in \mathcal{V}$, there exists Δ_V such that $\Delta \geqq \Delta_V$ implies that $J(F, \sigma_0, \Delta)$ is u.s. to $\int_{\sigma_0} F(d\sigma)$ with respect to V and that $J(F, \sigma_0, \Delta)$, $J(G, \sigma_0, \Delta)$ are summably equal within V. From this it follows that $J(G, \sigma_0, \Delta)$ is u.s. to $\int_{\sigma_0} F(d\sigma)$ with respect to $2V$ for $\Delta \geqq \Delta_V$. Since V is arbitrary $G(\sigma)$ is $\mathcal{V}$-integrable to $\int_{\sigma_0} F(d\sigma)$ by definition.

6.3. THEOREM. *If $F(\sigma)$, $G(\sigma)$ are $\mathcal{V}$-integrable on σ_0 to the same value, then they are d.e. on σ_0.*

6.4. COROLLARY. *If $F(\sigma)$ is $\mathcal{V}$-integrable on every $\sigma \subseteq \sigma_0$, then $F(\sigma)$ and $\int_\sigma F(d\sigma)$ are d.e. on σ_0.*

In view of the preceding results, one can characterize the (indefinite) $\mathcal{V}$-integral in terms of differential equivalence.

6.5. THEOREM. *In order that a function $I(\sigma)$ be the* (*indefinite*) *$\mathcal{V}$-integral of a given function $F(\sigma)$, it is both necessary and sufficient that it be closed* (*that is, $I(\sigma) = I(\sigma)_{cl}$*), *completely additive in the sense of Theorem* 5.5 *and d.e. to $F(\sigma)$ on each σ.*

6.6. COROLLARY. *In order that a single-valued function $I(\sigma)$ be the* (*indefinite*) *$\mathfrak{S}\,\mathcal{V}$-integral of a given function $F(\sigma)$, it is both necessary and sufficient that it be completely additive in the ordinary sense and d.e. to $F(\sigma)$ on each σ.*

7. **Transformation of an integrable function.** We now introduce a general type of linear transformation $T(X)$ defined on subsets of $\mathfrak{X}$ and whose values are sets in a similar space $\mathfrak{Y}$. The topology on $\mathfrak{Y}$ will be given by the system of sets U individual elements of which will be denoted by U. $T(X)$ will be subject to the following three conditions:

(1) $X_1 \subset X_2$ implies $T(X_1) \subset T(X_2)$.

(2) $T(X)$ is linear, that is, $T(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 T(X_1) + \alpha_2 T(X_2)$.

(3) $T(X)$ is continuous in the sense that $U \in U$ implies the existence of a $V_U \in \mathcal{V}$ such that $T(V_U) \subset U$.

The class of transformations described above contains as a special case the ordinary linear continuous point transformations $T(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$, where $T(X) = \{T(x) \mid x \in X\}$. It also contains the operation of forming the convex $C_o X$ of a set X and the operation of forming the "generalized convex" $C^*(X)$ of a set using the bounded generalized convex operators of G. B. Price [13, p. 7]. Observe that in these last two instances $\mathfrak{X} = \mathfrak{Y}$ and the transformations have the additional property of leaving individual points invariant.

7.1. THEOREM. *If $F(\sigma)$ is $\mathcal{V}$-integrable on σ_0, then the function $T(F(\sigma))$ is $\mathcal{V}$-integrable on σ_0 and $\int_{\sigma_0} T(F(d\sigma)) = T[\int_{\sigma_0} F(d\sigma)]_{cl}$.*

Since $T(X)$ is continuous, for arbitrary $U \in U$ there exists $V_U \in \mathcal{V}$ such that $T(V_U) \subset U$. Also, since $F(\sigma)$ is $\mathcal{V}$-integrable on σ_0, there exists Δ_U such that, if $\{\sigma_i\} = \Delta \geqq \Delta_U$, then there exists π_Δ for which $\pi \geqq \pi_\Delta$ implies

$$\sum_\pi F(\sigma_0 \cap \sigma_i) \subset \int_{\sigma_0} F(d\sigma) + V_U, \qquad \int_{\sigma_0} F(d\sigma) \subset \sum_\pi F(\sigma_0 \cap \sigma_i) + V_U.$$

Application of T to these relations and use of (1), (2) give

$$\sum_\pi T(F(\sigma_0 \cap \sigma_i)) \subset T\left[\int_{\sigma_0} F(d\sigma)\right] + U,$$

$$T\left[\int_{\sigma_0} F(d\sigma)\right] \subset \sum_\pi T(F(\sigma_0 \cap \sigma_i)) + U.$$

Therefore $J(T(F), \sigma_0, \Delta)$ is u.s. to $T[\int_{\sigma_0} F(d\sigma)]$ with respect to U, which completes the proof.

7.2. COROLLARY. *If $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0, then*[18] *$T(F(\sigma))$ is $\mathfrak{S}\,U$-integrable on σ_0 and $\int_{\sigma_0} F(T(d\sigma)) = T[\int_{\sigma_0} F(d\sigma)]$.*

7.3. COROLLARY. *If $\mathfrak{X} = \mathfrak{Y}$ and single elements are invariant under T, then $\mathfrak{S}\,\mathcal{V}$-integrability of $F(\sigma)$ implies that of $T(F(\sigma))$ and to the same value; that is, $\int_{\sigma_0} T(F(d\sigma)) = \int_{\sigma_0} F(d\sigma)$.*

8. **The $\mathfrak{S}\,\mathcal{V}$-integral in a complete space.** It will be recalled that the definition of the $\mathcal{V}$- and $\mathfrak{S}\,\mathcal{V}$-integrals (Definition 4.1) involves an assumption concerning the existence of the value of the integral in the space. This is, in part, necessitated by a lack of completeness in the space $\mathfrak{X}$. It is the purpose of this section to show that the existence assumption can be dropped in the case of the $\mathfrak{S}\,\mathcal{V}$-integral provided the space $\mathfrak{X}$ is complete relative to $\mathcal{V}$ (see §1 above).

8.1. DEFINITION. *The function $F(\sigma)$ is said to be conditionally $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0 if, for every $V \in \mathcal{V}$, there exists a $\Delta_{V\sigma_0}$ so that $\{\sigma_i^j\} = \Delta^j \geqq \Delta_{V\sigma_0}$ implies the existence of independent $\pi(\Delta^j)$ $(j = 1, 2)$ for which it is true that $\sum_{\pi_1} F(\sigma_0 \cap \sigma_i^1) - \sum_{\pi_2} F(\sigma_0 \cap \sigma_i^2) \subset V$ whenever $\pi_j \geqq \pi(\Delta^j)$ $(j = 1, 2)$.*

8.2. THEOREM. *If $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0, then $F(\sigma)$ is conditionally $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0.*

8.3. THEOREM. *If $\mathfrak{X}$ is complete relative to $\mathcal{V}$ and $F(\sigma)$ is conditionally $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0, then $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0.*

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[18] Observe that the continuity of $T(X)$ implies that single elements are carried into single elements; that is, $T(X)$ induces a linear continuous point transformation on $\mathfrak{X}$ to $\mathfrak{Y}$.

Let $\Delta_{V\sigma_0}=\{\sigma_{iV}\}$ be the subdivision and $\pi(\Delta_{V\sigma_0})=\pi_V$ the associated set of positive integers given by Definition 8.1. Denote by x_V a particular one of the elements in the set $\sum'_{\pi_V}F(\sigma_0\cap\sigma_{iV})$; then, by Definition 8.1, $J(F, \sigma_0, \Delta)$ is u.s. to x_V with respect to V for every $\Delta\geqq\Delta_{V\sigma_0}$. We now prove that $\{x_V\}$ is a fundamental $\mathcal{V}$-directed set.

Let $V\in\mathcal{V}$ be arbitrary and consider any pair of elements V_1, $V_2\in\mathcal{V}$ such that $V_i\subset V/2$ $(i=1, 2)$. For $\Delta\geqq\Delta_{V_1\sigma_0}\Delta_{V_2\sigma_0}$ we have that $J(F, \sigma_0, \Delta)$ is u.s. to x_{V_1} with respect to V_1 and to x_{V_2} with respect to V_2. It follows directly from this result that $x_{V_1}-x_{V_2}\in V_1+V_2\subset V$ and, hence, that $\{x_V\}$ is a fundamental $\mathcal{V}$-directed set. Let x_0 be the limit of this set. It remains to show that $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0 to the value x_0.

For arbitrary $V\in\mathcal{V}$ first chose $V_0\subset V/2$ such that $V'\subset V_0$ implies $\pm\{x_{V'}-x_0\}\in V/2$ and then choose $\Delta_{V_0\sigma_0}$ according to Definition 8.1. Then, if $\Delta\geqq\Delta_{V_0\sigma_0}$, $J(F, \sigma_0, \Delta)$ is u.s. to x_{V_0} with respect to V_0. But $\pm\{x_{V_0}-x_0\}\in V/2$; therefore $J(F, \sigma_0, \Delta)$ is u.s. to x_0 with respect to V; that is, $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on σ_0 to the value x_0.

The proof of the following lemma, though not difficult, is somewhat long; so will be omitted.

8.4. Lemma. *Conditional $\mathfrak{S}\,\mathcal{V}$-integrability on M implies conditional $\mathfrak{S}\,\mathcal{V}$-integrability on every σ.*

Combining Lemma 8.4 with Theorem 8.3 we have

8.5. Theorem[19]. *If $\mathfrak{X}$ is complete relative to $\mathcal{V}$ and $F(\sigma)$ is conditionally $\mathfrak{S}\,\mathcal{V}$-integrable on M, then $F(\sigma)$ is $\mathfrak{S}\,\mathcal{V}$-integrable on every σ.*

9. **A convergence theorem for the $\mathfrak{S}\,\mathcal{V}$-integral.** We consider only the $\mathfrak{S}\,\mathcal{V}$-integral in this section and restrict attention to integrable functions $F(\sigma)$ for which $\int_\sigma F(d\sigma)$ is absolutely continuous relative to a given, positive, completely additive measure function $m(\sigma)$. In view of Theorem 3.1, we could replace the above restriction by the stronger condition that $m(\sigma)=0$ imply $F(\sigma)=\theta$.

The following definition gives a generalization of the notion of approximate convergence[20] to functions $F(\sigma)$ of the type being considered here. It is also a generalization of a much stronger type of convergence used by Kolmogoroff [7, p. 665].

9.1. Definition. *A sequence of functions $\{F_n(\sigma)\}$ is said to converge approximately to $F(\sigma)$ relative to $m(\sigma)$ provided, for every integer n and $V\in\mathcal{V}$, there exists a $\sigma(n, V)\in\mathfrak{M}$ and a subdivision Δ_{nV} such that, for each V,*

([19]) Compare with Phillips' Theorem 4.1 [12, p. 122]. Observe that "completeness with respect to D" used by Phillips implies completeness relative to $\mathcal{V}$ (they are, in fact, equivalent; see Footnote 7).

([20]) See Definition 12.3 and Theorem 12.4 below.

$\lim_{n\to\infty} m(\sigma(n, V))=0$ *and, for* $\Delta \geqq \Delta_{nV}$, *it is true that* $J(F_n, \sigma, \Delta)$, $J(F, \sigma, \Delta)$ *are summably equal within* V *for every* $\sigma \subseteq M \cap C\sigma(n, V)$.

Observe that, if $\Delta=\{\sigma_i\}$ and $J(F_n, \sigma, \Delta)$, $J(F, \sigma, \Delta)$ are summably equal within V for arbitrary $\sigma \subseteq M \cap C\sigma(n, V)$, then $J(F_n, \sigma \cap \bigcup_\pi \sigma_i, \Delta)$, $J(F, \sigma \cap \bigcup_\pi \sigma_i, \Delta)$ are summably equal within V for *arbitrary* π and $\sigma \subseteq M \cap C\sigma(n, V)$. It follows immediately that $\sum_\pi F_n(\sigma \cap \sigma_i)$, $\sum_\pi F(\sigma \cap \sigma_i)$ are equal within V for *arbitrary* π and $\sigma \subseteq M \cap C\sigma(n, V)$. We thus obtain a result which is somewhat stronger than summable equality.

The proof of the next theorem will be omitted, since it is essentially contained in the first part of the proof of Theorem 9.5 below.

9.2. THEOREM. *Let* $F_n(\sigma)$ *be* $\mathfrak{S}\mathcal{V}$*-integrable on every* σ *and let* $\int_\sigma F_n(d\sigma)$ *be absolutely continuous relative to* $m(\sigma)$ $(n=0, 1, 2, \cdots)$. *Then, if* $F_n(\sigma)$ *converges approximately to* $F_0(\sigma)$ *relative to* $m(\sigma)$, *the following are equivalent*:

(i) $\lim_{n\to\infty} \int_\sigma F_n(d\sigma)=\int_\sigma F_0(d\sigma)$ *uniformly in* σ.

(ii) $\int_\sigma F_n(d\sigma)$ *are equi-absolutely continuous relative to* $m(\sigma)$.

9.3. DEFINITION. *The function* $F(\sigma)$ *is said to be* $\mathfrak{S}\mathcal{V}$*-integrable uniformly in* σ *provided* $F(\sigma)$ *is* $\mathfrak{S}\mathcal{V}$*-integrable on every* σ *and, for each* $V \in \mathcal{V}$, *there exists* Δ_V *independent of* σ *such that, if* $\Delta \geqq \Delta_V$, *then* $J(F, \sigma, \Delta)$ *is u.s. to* $\int_\sigma F(d\sigma)$ *with respect to* V *uniformly in* σ.

9.4. LEMMA. *Let* $F(\sigma)$ *be* $\mathfrak{S}\mathcal{V}$*-integrable uniformly in* σ *and let* σ_0 *be such that* $\pm\int_{\sigma\cap\sigma_0} F(d\sigma) \in V$ *for all* σ. *Then there exists* Δ_V *such that* $\{\sigma_i\}=\Delta \geqq \Delta_V$ *implies* $\pm\sum_{\pi'} F(\sigma_0 \cap \sigma_i) \subset 2V$ *for arbitrary* π'.

From the definition of uniform $\mathfrak{S}\mathcal{V}$-integrability, there exists Δ_V such that $\{\sigma_i\}=\Delta \geqq \Delta_V$ implies that $J(F, \sigma, \Delta)$ is u.s. to $\int_\sigma F(d\sigma)$ with respect to V uniformly in σ, that is, there exists π_Δ independent of σ such that, if $\pi \geqq \pi_\Delta$,

$$\pm\left\{ \sum_\pi F(\sigma \cap \sigma_i) - \int_\sigma F(d\sigma)\right\} \subset V. \tag{1}$$

Now let π' be arbitrary and set $\sigma=\sigma_0 \cap (\bigcup_{\pi'} \sigma_i)$, $\pi=\pi' \cup \pi_\Delta$ in (1). Since $\pm\int_{\sigma\cap\sigma_0} F(d\sigma) \in V$ for all σ, this completes the proof.

9.5. THEOREM. *Let* $\mathfrak{X}$ *be sequentially complete,* $F_n(\sigma)$ $\mathfrak{S}\mathcal{V}$*-integrable uniformly in* σ, *and* $\int_\sigma F_n(d\sigma)$ *absolutely continuous relative to* $m(\sigma)$ $(n=1, 2, \cdots)$. *Then, if* $\{F_n(\sigma)\}$ *converges approximately to* $F(\sigma)$ *relative to* $m(\sigma)$, *the following are equivalent*:

(i) $F(\sigma)$ *is* $\mathfrak{S}\mathcal{V}$*-integrable uniformly in* σ *and* $\lim_{n\to\infty} \int_\sigma F_n(d\sigma)=\int_\sigma F(d\sigma)$ *uniformly in* σ.

(ii) $\lim_{n\to\infty} \int_\sigma F_n(d\sigma)$ *exists for every* σ.

(iii) $\int_\sigma F_n(d\sigma)$ *are equi-absolutely continuous relative to* $m(\sigma)$.

That (i) implies (ii) is trivial and (ii) implies (iii) by Theorem 3.2. We prove that (iii) implies (i).

Let $V \in \mathcal{V}$ be arbitrary and set $\sigma_{mn} = \sigma(m, V) \cup \sigma(n, V)$, where $\sigma(m, V)$, $\sigma(n, V)$ are given by Definition 9.1. If $\Delta \geqq \Delta_{mV} \cdot \Delta_{nV}$, where Δ_{mV}, Δ_{nV} are given by Definition 9.1, and if $\sigma \subseteq M \cap C\sigma_{mn}$, then $J(F_m, \sigma, \Delta)$, $J(F_n, \sigma, \Delta)$ are each summably equal to $J(F, \sigma, \Delta)$ within V. It follows that $J(F_m, \sigma, \Delta)$, $J(F_n, \sigma, \Delta)$ are summably equal within $2V$. Now an application of Corollary 5.9 gives

$$\int_{\sigma \cap C\sigma_{mn}} F_m(d\sigma) - \int_{\sigma \cap C\sigma_{mn}} F_n(d\sigma) \in (2V)_{cl} \subset 3V.$$

This holds for arbitrary σ and all m, n. Using (iii) ,we obtain n_V such that $m, n \geqq n_V$ implies $\pm \int_{\sigma \cap \sigma_{mn}} F_k(d\sigma) \in V$ for arbitrary σ and all k. Therefore, if $m, n \geqq n_V$,

$$\int_\sigma F_m(d\sigma) - \int_\sigma F_n(d\sigma) \in 3V + 2V \subset 6V,$$

where σ is arbitrary and n_V obviously does not depend on σ. It follows that $\{\int_\sigma F_n(d\sigma)\}$ is a fundamental sequence uniformly in σ. Since $\mathfrak{X}$ is sequentially complete, there exists $I(\sigma) \in \mathfrak{X}$ such that $\lim_{n\to\infty} \int_\sigma F_n(d\sigma) = I(\sigma)$ uniformly in σ. It remains to show that $F(\sigma)$ is $\mathfrak{S}\mathcal{V}$-integrable uniformly in σ to the value $I(\sigma)$.

Let $V \in \mathcal{V}$ be arbitrary and select a subsequence of the $F_n(\sigma)$, which we continue to denote by $\{F_n(\sigma)\}$, having the following two properties:

(a) $\pm\{\int_\sigma F_1(d\sigma) - I(\sigma)\} \in 2^{-3}V$ for all σ.

(b) There exist $\tau_n \in \mathfrak{M}$ such that[21] $m(\tau_n) < \delta(2^{-n-4}V)$, $\tau_n \supset \tau_{n+1}$, and also there exist Δ_{nV} such that $\Delta \geqq \Delta_{nV}$ implies $J(F_n, \sigma, \Delta)$, $J(F, \sigma, \Delta)$ are summably equal within $\pm 2^{-n-2}V$ for all $\sigma \subseteq M \cap C\tau_n$.

Set $\sigma_1^0 = M \cap C\tau_1$ and $\sigma_n^0 = \tau_{n-1} \cap C\tau_n$ for $n \geqq 2$, and consider the subdivision $\Delta^0 = \{\sigma_i^0\}$. Since $m(\sigma \cap \sigma_n^0) < \delta(2^{-n-3}V)$ (for $n \geqq 2$), it follows that $\pm \int_{\sigma \cap \sigma_n^0} F_n(d\sigma) \in 2^{-n-3}V$ for arbitrary σ. Hence, by Lemma 9.4, there exists $\Delta^n \geqq \Delta^0 \Delta_{nV}$ such that $\{\sigma_i\} = \Delta \geqq \Delta^n$ implies

$$\pm \sum_{\pi'} F_n(\sigma \cap \sigma_n^0 \cap \sigma_i) \subset V/2^{n+2},$$

for arbitrary π', σ and $n \geqq 2$. Out of property (b) and the remark following Definition 9.1, it follows that $\sum_{\pi'} F_n(\sigma \cap \sigma_n^0 \cap \sigma_i)$, $\sum_{\pi'} F(\sigma \cap \sigma_n^0 \cap \sigma_i)$ are equal within $\pm 2^{-n-2}V$ and, hence, that

$$\pm \sum_{\pi'} F(\sigma \cap \sigma_n^0 \cap \sigma_i) \subset V/2^{n+1}.$$

This result holds for arbitrary $n \geqq 2$, σ, π' and $\{\sigma_i\} = \Delta \geqq \Delta^n$.

Now define Δ^1 such that $\Delta \geqq \Delta^1$ implies $J(F, \sigma, \Delta)$ u.s. to $\int_\sigma F_1(d\sigma)$ with respect to $2^{-4}V$ uniformly in σ, and let Δ_V be the sum of the subdivisions Δ^n

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[21] $\delta(eV) > 0$ is chosen so that, if $m(\sigma) < \delta(eV)$, then $\pm \int_\sigma F_k(d\sigma) \in eV$ for all k.

($n=1, 2, \cdots$) over the subdivision Δ^0 (see §4 above). For $\{\sigma_i\}=\Delta\geqq\Delta_V$ and arbitrary π', σ, we have

$$(2)\qquad \pm\sum_{\pi'}F(\sigma\cap\tau_1\cap\sigma_i)=\pm\sum_{n=2}^{\infty}\sum_{\pi'}F(\sigma\cap\overset{0}{\sigma_n}\cap\sigma_i)\subset\sum_{n=2}^{N_{\pi'}}V/2^{n+1}\subset V/2,$$

where $N_{\pi'}$ is the largest n for which $\sigma_n^0\cap(\bigcup_{\pi'}\sigma_i)\neq 0$.

If $\Delta\geqq\Delta_V$, then $J(F_1, \sigma\cap\sigma_1^0, \Delta)$ is u.s. to $\int_{\sigma\cap\sigma^0}F_1(d\sigma)$ with respect to $2^{-4}V$ uniformly in σ. Moreover, since $m(\tau_1)<\delta(2^{-5}V)$, it follows that $\pm\int_{\sigma\cap\tau_1}F_1(d\sigma)\in 2^{-5}V$ and, hence, that $J(F_1, \sigma\cap\sigma_1^0, \Delta)$ is u.s. to $\int_\sigma F_1(d\sigma)$ with respect to $2^{-3}V$ uniformly in σ. Applying (a), we obtain $J(F_1, \sigma\cap\sigma_1^0, \Delta)$ u.s. to $I(\sigma)$ with respect to $V/2$ and, applying (b) again, we have $J(F, \sigma\cap\sigma_1^0, \Delta)$ u.s. to $I(\sigma)$ with respect to $V/2$ uniformly in σ. From this last result and (2) it follows that $J(F, \sigma, \Delta)$ is u.s. to $I(\sigma)$ with respect to V uniformly in σ, which completes the proof of Theorem 9.5.

Part III. Integration with respect to a "bilinear" function[22]

10. **The $\mathcal{V}_B$-integral.** Let $\mathfrak{X}$, as usual, be a convex linear topological space and let $\mathfrak{Y}$ be simply a linear space. We introduce a "bilinear" function $B[y, \sigma]$ subject to the following four conditions:

B1. *For every $y\in\mathfrak{Y}$ and*[23] $\sigma\in\mathfrak{M}$, *$B[y, \sigma]$ is a unique element of $\mathfrak{X}$.*

B2. *$B[y, \sigma]$ is linear (not necessarily continuous) in y for each σ; that is, $B[\alpha_1y_1+\alpha_2y_2, \sigma]=\alpha_1B[y_1, \sigma]+\alpha_2B[y_2, \sigma]$.*

B3. *For each y, $B[y, \sigma]$ is a completely additive function of σ.*

B4. *There exists a real number $\beta\geqq 1$ such that*[24]

$$\sum_{i=1}^{m}B[Y_i, \sigma_i]\subset V\quad\textit{implies}\quad\sum_{i=1}^{m}\sum_{j=1}^{n_i}B[Y_i, \sigma_i^j]\subset\beta V,$$

where $Y_i\subset\mathfrak{Y}$, $\sigma_i\cap\sigma_j=0$ ($i\neq j$), $\sigma_i=\bigcup_{j=1}^{n_i}\sigma_i^j$, $\sigma_i^j\cap\sigma_i^k=0$ ($j\neq k$).

The functions $y(\sigma)$ to be considered in this part will be multi-valued and defined on $\mathfrak{M}$ to $\mathfrak{Y}$. They will be subject without exception to the restriction that $\sigma_1\subseteq\sigma_2$ shall imply $y(\sigma_1)\subseteq y(\sigma_2)$. Such functions are described as *contractive*[25]. If $\Delta=\{\sigma_i\}$ is an arbitrary subdivision, the sequence of sets $\{B[y(\sigma\cap\sigma_i), \sigma\cap\sigma_i]\}$ will be denoted by the symbol $J_B(y, \sigma, \Delta)$.

10.1. Definition. *$y(\sigma)$ is said to be $\mathcal{V}_B$-integrable on σ_0 provided there exists an element $I_B(y, \sigma_0)\in\mathfrak{X}$ such that, for every $V\in\mathcal{V}$, there exists a Δ_{σ_0V} for which $J_B(y, \sigma_0, \Delta_{\sigma_0V})$ is u.s. to $I_B(y, \sigma_0)$ with respect to V. $I_B(y, \sigma_0)$ is the value of the integral and we write $I_B(y, \sigma_0)=\int_{\sigma_0}B[y, d\sigma]$.*

(22) The general idea of considering integration with respect to a "bilinear" function was suggested by T. H. Hildebrandt. This paper represents a development from that idea.

(23) See Footnote 14 above.

(24) If $Y\subset\mathfrak{Y}$, then $B[Y, \sigma]=\{B[y, \sigma]\mid y\in Y\}$.

(25) Observe that, if $y(t)$ is a point function, the associated set function $y(\sigma)=\{y(t)\mid t\in\sigma\}$ is contractive.

Observe that this definition is weaker in form than Definition 4.1, because the assumption here is that $J_B(y, \sigma_0, \Delta)$ be u.s. to $I_B(y, \sigma_0)$ only for $\Delta = \Delta_{\sigma_0 V}$ rather than $\Delta \geqq \Delta_{\sigma_0 V}$. The weakening of the definition of integrability is balanced by the conditions on $B[y, \sigma]$. Definition 10.1 is essentially that used by R. S. Phillips [12, p. 118] and the integral obtained here will be seen to reduce to his as a special case (Theorem 15.3).

10.2. LEMMA. *Let* $Y_i \subset \mathfrak{Y}$, $\sigma_i \cap \sigma_j = 0$ $(i \neq j)$, $\sigma_i = \bigcup_{j=1}^{\infty} \sigma_i^j$, $\sigma_i^j \cap \sigma_i^k = 0$ $(j \neq k)$; *then*

$$\pm \left\{ x + \sum_{i=1}^{m} B[Y_i, \sigma_i] \right\} \subset V$$

implies the existence of a π_0 *independent of* x *such that, if* $\pi_i \geqq \pi_0$,

$$\pm \left\{ x + \sum_{i=1}^{m} \sum_{j \in \pi_i} B[Y_i, \sigma_i^j] \right\} \subset 4\beta V.$$

We have immediately that

$$\sum_{i=1}^{m} B[Y_i - Y_i, \sigma_i] \subset 2V;$$

hence, by condition B4,

$$\sum_{i=1}^{m} \left\{ \sum_{j \in \pi_i} B[Y_i - Y_i, \sigma_i^j] + B\left[Y_i - Y_i, \bigcup_{j \notin \pi_i} \sigma_i^j \right] \right\} \subset 2\beta V,$$

where the π_i are completely arbitrary. This can be written in the form

$$\sum_{i=1}^{m} \sum_{j \in \pi_i} B[Y_i, \sigma_i^j] + \sum_{i=1}^{m} B\left[Y_i, \bigcup_{j \notin \pi_i} \sigma_i^j \right] - \sum_{i=1}^{m} \sum_{j \in \pi_i} B[Y_i, \sigma_i^j] - \sum_{i=1}^{m} B\left[Y_i, \bigcup_{j \notin \pi_i} \sigma_i^j \right] \subset 2\beta V. \tag{1}$$

This gives

$$\pm \left\{ \sum_{i=1}^{m} \sum_{j \in \pi_i} B[Y_i, \sigma_i^j] + \sum_{i=1}^{m} B\left[Y_i, \bigcup_{j \notin \pi_i} \sigma_i^j \right] - \sum_{i=1}^{m} B[Y_i, \sigma_i] \right\} \subset 2\beta V.$$

It follows by the hypothesis of the lemma that

$$\pm \left\{ x + \sum_{i=1}^{m} \sum_{j \in \pi_i} B[Y_i, \sigma_i^j] + \sum_{i=1}^{m} B\left[Y_i, \bigcup_{j \notin \pi_i} \sigma_i^j \right] \right\} \subset 2\beta V + V. \tag{2}$$

Now let y_i be some particular element of Y_i; then from condition B3 it is evident that there exists a π_0 such that $\pi_i \geqq \pi_0$ implies

$$\pm \sum_{i=1}^{m} B\left\{ y_i, \bigcup_{j \notin \pi_i} \sigma_i^j \right\} \in V. \tag{3}$$

Relations (2) and (3) together give

$$\pm \left\{ x + \sum_{i=1}^{m} \sum_{j\in\pi_i} B[Y_i, \sigma_i^j] \right\} \subset 2\beta V + V + V \subset 4\beta V,$$

which completes the proof.

10.3. LEMMA. *If $J_B(y, \sigma, \Delta_0)$ is u.s. to $I_B(y, \sigma)$ with respect to V, then $J_B(y, \sigma, \Delta)$ is u.s. to $I_B(y, \sigma)$ with respect to $7\beta V$ for all $\Delta \geqq \Delta_0$.*

There is no loss in taking $\sigma = M$. Let $\Delta_0 = \{\sigma_i\}$; then by hypothesis there exists π_0 such that $\pi \geqq \pi_0$ gives

$$\pm \left\{ \sum_{\pi} B[y(\sigma_i), \sigma_i] - I_B(y, M) \right\} \subset V. \tag{1}$$

Consider $\{\sigma_i^j\} = \Delta \geqq \Delta_0$, where $\sigma_i = \bigcup_{j=1}^{\infty} \sigma_i^j$. By Lemma 10.2 there exists π_0' such that $\pi_i \geqq \pi_0'$ implies

$$\pm \left\{ \sum_{i\in\pi_0} \sum_{j\in\pi_i} B[y(\sigma_i^j), \sigma_i^j] - I_B(y, M) \right\} \subset 4\beta V. \tag{2}$$

Now let ν_0 denote those integer pairs (i, j) for which $i \in \pi_0$ and $j \in \pi_0'$, and consider any finite set of integer pairs ν which contains ν_0; that is, $\nu \geqq \nu_0$. Set $\nu' = \{(i, j) \mid (i, j) \in \nu, i \in \pi_0\}$ and $\nu'' = \{(i, j) \mid (i, j) \in \nu, i \notin \pi_0\}$. It follows from (2) that

$$\pm \left\{ \sum_{\nu'} B[y(\sigma_i^j), \sigma_i^j] - I_B(y, M) \right\} \subset 4\beta V. \tag{3}$$

Moreover, from (1) we have

$$\pm \sum_{\pi'} B[y(\sigma_i), \sigma_i] \subset 2V, \tag{4}$$

for arbitrary π' such that $\pi' \cap \pi_0 = 0$. Because of the arbitrary character of π' in (4), we can write

$$\pm \sum_{\pi'} B[y(\sigma_i) \cup \theta, \sigma_i] \subset 2V. \tag{5}$$

Now let $\pi' = \{i \mid (i, j) \in \nu'' \text{ for some } j\}$, $\pi_i' = \{j \mid (i, j) \in \nu''\}$ and apply B4 to (5) to obtain

$$\pm \left\{ \sum_{\nu''} B[y(\sigma_i) \cup \theta, \sigma_i^j] + \sum_{\pi'} B\left[y(\sigma_i) \cup \theta, \bigcup_{j \notin \pi_i'} \sigma_i^j \right] \right\} \subset 2\beta V.$$

From this it follows that

$$\pm \sum_{\nu''} B[y(\sigma_i^j), \sigma_i^j] \subset 2\beta V. \tag{6}$$

Combining (3) and (6) we obtain

$$\pm \left\{ \sum_{\nu} B[y(\sigma_i^j), \sigma_i^j] - I_B(y, M) \right\} \subset 4\beta V + 2\beta V \subset 7\beta V.$$

As a consequence of Lemma 10.3 we have

10.4. THEOREM. *$\mathcal{V}_B$-integrability of $y(\sigma)$ is equivalent to $\mathfrak{S}\,\mathcal{V}$-integrability of $F(\sigma) = B[y(\sigma), \sigma]$.*

Investigation of the form in which the basic properties of the $\mathfrak{S}\mathcal{V}$-integral appear in the special case of the $\mathcal{V}_B$-integral will be left to the reader.

The following lemma is in preparation for the proof that $\mathcal{V}_B$-integrability of $y(\sigma)$ for every σ implies $\mathfrak{S}\,\mathcal{V}$-integrability of $F(\sigma)$ uniformly in σ.

10.5. LEMMA. *If for a given $\Delta = \{\sigma_i\}$ there exists π_Δ such that $\pi_i \geqq \pi_\Delta$ $(i = 1, 2)$ implies*

$$(1) \qquad \sum_{\pi_1} B[y(\sigma_i), \sigma_i] - \sum_{\pi_2} B[y(\sigma_i), \sigma_i] \subset V,$$

then for arbitrary σ and $\pi_i \geqq \pi_\Delta$ $(i = 1, 2)$

$$\sum_{\pi_1} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] - \sum_{\pi_2} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] \subset 4\beta V.$$

Taking $\pi_1 = \pi_2 = \pi_\Delta$, we obtain from (1)

$$\sum_{\pi_\Delta} B[y(\sigma_i) - y(\sigma_i), \sigma_i] \subset V.$$

An application of B4 gives

$$\sum_{\pi_\Delta} \{B[y(\sigma_i) - y(\sigma_i), \sigma \cap \sigma_i] - B[y(\sigma_i) - y(\sigma_i), \sigma_i \cap C\sigma]\} \subset \beta V.$$

Since $\theta \in y(\sigma_i) - y(\sigma_i)$, it follows that

$$(2) \qquad \sum_{\pi_\Delta} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] - \sum_{\pi_\Delta} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] \subset \beta V.$$

Again from (1) we have for arbitrary $\pi' \cap \pi_\Delta = 0$

$$\pm \sum_{\pi'} B[y(\sigma_i), \sigma_i] \subset V,$$

which, because of the arbitrary character of π', implies (as in the proof of Lemma 10.3)

$$\pm \sum B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] \subset \beta V.$$

Combining this last result with (2) completes the proof of the lemma.

10.6. THEOREM. *$\mathcal{V}_B$-integrability of $y(\sigma)$ on every σ is equivalent to $\mathfrak{S}\mathcal{V}$-integrability of $F(\sigma) = B[y(\sigma), \sigma]$ uniformly in σ.*

Because of Theorem 10.4, for every $V \in \mathcal{V}$ there exists Δ_V such that, if $\{\sigma_i\} = \Delta \geqq \Delta_V$, then there exists π_Δ for which $\pi \geqq \pi_\Delta$ implies

$$\pm \left\{ \sum_{\pi} B[y(\sigma_i), \sigma_i] - \int_M B[y, d\sigma] \right\} \subset V/2.$$

Therefore, if $\pi_i \geqq \pi_\Delta$ $(i = 1, 2)$,

$$\sum_{\pi_1} B[y(\sigma_i), \sigma_i] - \sum_{\pi_2} B[y(\sigma_i), \sigma_i] \subset V.$$

From Lemma 10.5 it follows that

$$(1) \qquad \pm \left\{ \sum_{\pi_1} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] - \sum_{\pi_2} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] \right\} \subset 4\beta V$$

for arbitrary σ and $\pi_i \geqq \pi_\Delta$ $(i = 1, 2)$. Now, for a particular σ choose $\Delta_{\sigma V}$ such that $\Delta' \geqq \Delta_{\sigma V}$ implies $J_B(y, \sigma, \Delta')$ u.s. to $\int_\sigma B[y, d\sigma]$ with respect to V. Let $\{\sigma_i'\} = \Delta' = \Delta\Delta_{\sigma V}$, then it follows that there exists $\pi_{\Delta'}$ such that, if $\pi' \geqq \pi_{\Delta'}$,

$$(2) \qquad \pm \left\{ \sum_{\pi'} B[y(\sigma \cap \sigma_i'), \sigma \cap \sigma_i'] - \int_\sigma B[y, d\sigma] \right\} \subset V.$$

Now in (1) choose $\pi_2 \geqq \pi_\Delta$ such that $\bigcup_{\pi_{\Delta'}} \sigma_i' \subset \bigcup_{\pi_2} \sigma_i$. Then, since $\Delta' \geqq \Delta$, we can apply Lemma 10.2 to (1) and obtain the existence of a $\pi_2' \geqq \pi_{\Delta'}$ such that

$$\pm \left\{ \sum_{\pi_1} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] - \sum_{\pi_2} B[y(\sigma \cap \sigma_i'), \sigma \cap \sigma_i'] \right\} \subset 16\beta^2 V.$$

This result with (2) yields

$$\pm \left\{ \sum_{\pi} B[y(\sigma \cap \sigma_i), \sigma \cap \sigma_i] - \int_\sigma B[y, d\sigma] \right\} \subset 18\beta^2 V,$$

for arbitrary $\pi \geqq \pi_\Delta$. Since π_Δ does not depend on σ, the proof is complete.

11. **The $\mathcal{V}_B$-integral in a complete space.** It has already been observed that the definition of the $\mathcal{V}_B$-integral is weaker in form than the definition of the $\mathfrak{S}\,\mathcal{V}$-integral. Similarly, conditional $\mathcal{V}_B$-integrability can be defined in a weaker form than conditional $\mathfrak{S}\,\mathcal{V}$-integrability.

11.1. DEFINITION. *$y(\sigma)$ is said to be conditionally $\mathcal{V}_B$-integrable on σ_0 provided for every $V \in \mathcal{V}$ there exists $\Delta_{\sigma_0 V} = \{\sigma_i\}$ and $\pi_{\sigma_0 V}$ such that, if $\pi_i \geqq \pi_{\sigma_0 V}$ $(i = 1, 2)$, then*

$$\sum_{\pi_1} B[y(\sigma_0 \cap \sigma_i), \sigma_0 \cap \sigma_i] - \sum_{\pi_2} B[y(\sigma_0 \cap \sigma_i), \sigma_0 \cap \sigma_i] \subset V.$$

The following theorem follows easily from Lemma 10.2.

11.2. THEOREM. *Conditional $\mathcal{V}_B$-integrability of $y(\sigma)$ is equivalent to conditional $\mathfrak{S}\,\mathcal{V}$-integrability of $F(\sigma) = B[y(\sigma), \sigma]$.*

Out of Theorems 11.2 and 8.5 we obtain

11.3. THEOREM. *If $\mathfrak{X}$ is complete relative to $\mathcal{V}$ and $y(\sigma)$ is conditionally $\mathcal{V}_B$-integrable on M, then $y(\sigma)$ is $\mathcal{V}_B$-integrable on every σ.*

12. Absolute continuity and a convergence theorem for the $\mathcal{V}_B$-integral. In this section we assume given a positive completely additive measure function $m(\sigma)$ defined over $\mathfrak{M}$ such that $m(\sigma)=0$ implies $B[y, \sigma]=\theta$ for every $y\in\mathfrak{Y}$. We have immediately that $m(\sigma)=0$ implies $\int_\sigma B[y, d\sigma]=\theta$, where $y(\sigma)$ is arbitrary. This remark plus Theorem 10.4, Corollary 5.6 and Theorem 3.1 enables us to state

12.1. THEOREM. *If $y(\sigma)$ is $\mathcal{V}_B$-integrable on every σ, then $\int_\sigma B[y, d\sigma]$ is completely additive and absolutely continuous relative to $m(\sigma)$.*

Throughout this and the following section we assume a topology(26) on the space $\mathfrak{Y}$ given by the system of neighborhoods $\mathcal{U}$ individual elements of which will be denoted by U. Also $B[y, \sigma]$ will be subject to the following condition in addition to B1–B4.

B5. *$B[y, M]$ is continuous on $\mathfrak{Y}$ to $\mathfrak{X}$; that is, for every $V\in\mathcal{V}$ there exists $U_V\in U$ such that $B[U_V, M]\subset V$.*

12.2. THEOREM. *If $B[y, \sigma]$ satisfies* B1–B5, *then $B[y, \sigma]$ is continuous for each σ and uniformly in σ.*

Let $V\in\mathcal{V}$ and choose U_V such that $B[U_V, M]\subset V/\beta$. Applying B4 we get $B[U_V, \sigma]+B[U_V, M\cap C\sigma]\subset V$, where σ is arbitrary. Since $\theta\in U_V$, we have $B[U_V, \sigma]\subset V$. But U_V does not depend on σ; hence $B[y, \sigma]$ is continuous uniformly in σ.

12.3. DEFINITION. *The sequence of functions $\{y_n(\sigma)\}$ is said to converge approximately*(27) *to $y(\sigma)$ relative to $m(\sigma)$ provided, for every n, U, there exists $\sigma(n, U)\in\mathfrak{M}$ such that* $\lim_{n\to\infty} m(\sigma(n, U))=0$ *for each U and for $\sigma\subseteq M\cap C\sigma(n, U)$ it is true that the sets $y_n(\sigma)$, $y(\sigma)$ are equal within U.*

12.4. THEOREM. *If $y_n(\sigma)$ converges approximately to $y(\sigma)$ according to Definition* 12.3, *then $F_n(\sigma)=B[y_n(\sigma), \sigma]$ converges approximately to $F(\sigma)=B[y(\sigma), \sigma]$ according to Definition* 9.1.

Given $V\in\mathcal{V}$, because of Theorem 12.2 we can choose U_V such that $B[U_V, \sigma]\subset V/\beta$ for every σ. It follows immediately from B4 that $\sum_{i=1}^m B[U_V, \sigma_i]\subset V$ for arbitrary disjoint σ_i. Now, if $\sigma_i\subseteq M\cap C\sigma(n, U_V)$, then

$$y_n(\sigma_i) \subset y(\sigma_i) + U_V.$$

Applying $B[y, \sigma_i]$ to this relation and adding, we obtain

(26) $\mathfrak{Y}$ is not necessarily assumed to be convex.

(27) This definition is a generalization of the one used by Phillips [12, p. 125].

$$\sum_{i=1}^{m} B[y_n(\sigma_i), \sigma_i] \subset \sum_{i=1}^{m} B[y(\sigma_i), \sigma_i] + V.$$

In a similar manner we obtain

$$\sum_{i=1}^{m} B[y(\sigma_i), \sigma_i] \subset \sum_{i=1}^{m} B[y_n(\sigma_i), \sigma_i] + V.$$

It follows from these relations that $J_B(y_n, \sigma, \Delta)$, $J_B(y, \sigma, \Delta)$ are summably equal within V for *every* Δ and $\sigma \subseteq M \cap C\sigma(n, U_V)$. Since $\lim_{n\to\infty} m(\sigma(n, U_V)) = 0$, the proof is complete.

Collecting the results of Theorems 10.6, 12.4, 9.5, we can state

12.5. THEOREM. *Let $\mathfrak{X}$ be sequentially complete, $y_n(\sigma)$ be $\mathcal{V}_B$-integrable on each σ ($n=1, 2, \cdots$), and $y_n(\sigma)$ converge approximately to $y(\sigma)$. Then the following are equivalent:*

(i) *$y(\sigma)$ is $\mathcal{V}_B$-integrable on each σ and* $\lim_{n\to\infty} \int_\sigma B[y_n, d\sigma] = \int_\sigma B[y, d\sigma]$ *uniformly in σ.*

(ii) $\lim_{n\to\infty} \int_\sigma B[y_n, d\sigma]$ *exists for each σ.*

(iii) $\int_\sigma B[y_n, d\sigma]$ *are equi-absolutely continuous relative to $m(\sigma)$.*

13. **Measurable functions. An existence theorem for the $\mathcal{V}_B$-integral.** The following definition of measurability for functions $y(\sigma)$ of the type being considered here is a generalization of a definition given by Price [13, p. 25] for single-valued point functions with values in a Banach space. We are interested here only in generalizing Theorem (16.1) of [13] to the $\mathcal{V}_B$-integral, where $B[y, \sigma]$ is subject to all of the five conditions B1–B5.

13.1. DEFINITION. *The function $y(\sigma)$ is said to be measurable ($\mathfrak{M}$) on the set σ_0 provided, for every set Y dense in $y(\sigma_0)$, $y \in Y$ and $U \in \mathcal{V}$ implies the existence of $\sigma_{yU} \in \mathfrak{M}$ such that $y(\sigma_{yU}) \subset y + U$ and such that $\sigma_0 = \bigcup_{y\in Y} \sigma_{yU}$.*

The next definition gives a generalization of a familiar condition frequently imposed on Lebesgue measurable functions to insure the existence of a finite Lebesgue integral.

13.2. DEFINITION. *The function $y(\sigma)$ is said to be B-summable provided, for every $V \in \mathcal{V}$, there exists Δ_V such that, if $\{\sigma_i\} = \Delta \geqq \Delta_V$, then there exists π_Δ for which $\pi \cap \pi_\Delta = 0$ implies $\pm\sum_\pi B[y(\sigma_i), \sigma_i] \subset V$.*

Observe that, if $y(\sigma)$ is B-summable, then $J_B(y, M, \Delta)$ is u.s. to $\sum_{\pi_\Delta} B[y(\sigma_i), \sigma_i]$ with respect to V for $\Delta \geqq \Delta_V$. Also it can be proved that $y(\sigma)$ is B-summable if there exists Δ_V with the property that, for each $\Delta \geqq \Delta_V$, there is a *bounded*(28) set I_Δ such that $J_B(y, M, \Delta)$ is u.s. to I_Δ with respect to V.

(28) A set $X \subset \mathfrak{X}$ is said to be *bounded* provided, for every $V \in \mathcal{V}$, there exists $\alpha > 0$ such that $X \subset \alpha V$ [11].

A function $y(\sigma)$ is said to be *separable* provided the set $y(M)$ is separable. *Almost separable* (B) will mean that there exists a set σ_0 of B-measure zero (that is[29], $B[y, \sigma_0]=\theta$ for every $y\in\mathfrak{Y}$) such that $y(M\cap C\sigma_0)$ is separable [13, p. 25].

13.3. THEOREM. *Let $\mathfrak{X}$ be complete relative to $\mathcal{U}$ and let $y(\sigma)$ be B-summable, almost separable (B) and measurable $(\mathfrak{M})$ on the set $M\cap C\sigma_0$, where σ_0 is the set of B-measure zero such that $y(M\cap C\sigma_0)$ is separable. Then $y(\sigma)$ is $\mathcal{U}_B$-integrable on each σ.*

It may as well be assumed at the outset that $y(\sigma)$ is separable. Also, in view of Theorem 11.3, it will be sufficient to prove $y(\sigma)$ conditionally $\mathcal{U}_B$-integrable on M.

Let $V\in\mathcal{U}$ be arbitrary and choose $U_V\in\mathcal{U}$ so that $B[U_V, \sigma]\subset V/\beta$ for all σ. Then, for arbitrary disjoint σ_i, it follows by condition B4 that

$$\sum_{i=1}^{m} B[U_V, \sigma_i]\subset V. \tag{1}$$

Since $y(\sigma)$ is measurable $(\mathfrak{M})$ there exists $\sigma_n^0\in\mathfrak{M}$ such that $y(\sigma_n^0)\subset y_n+U'$ and $M=\bigcup_{n=1}^{\infty}\sigma_n^0$, where $\{y_n\}$ is the separating sequence for $y(M)$ and $U'\in\mathcal{U}$ is chosen so that $\pm 2U'\subset U$. Let Δ_V be the subdivision given by Definition 13.2 and choose $\{\sigma_i\}=\Delta_V'\geqq\Delta_V$ such that each σ_i is contained in one of the sets σ_n^0. Observe that $y(\sigma_i)-y(\sigma_i)\subset U_V$ for every i.

Since $\{\sigma_i\}\geqq\Delta_V$, there exists π_V such that $\pi'\cap\pi_V=0$ implies $\pm\sum_{\pi'}B[y(\sigma_i), \sigma_i]\subset V$. For arbitrary $\pi_i\geqq\pi_V$ $(i=1, 2)$ set $\pi_i=\pi_V\cup\pi_i'$ where $\pi_i'\cap\pi_V=0$. Then

$$\begin{aligned}\sum_{\pi_1} B[y(\sigma_i), \sigma_i] &- \sum_{\pi_2} B[y(\sigma_i), \sigma_i] \\ &= \sum_{\pi_V} B[y(\sigma_i) - y(\sigma_i), \sigma_i] + \sum_{\pi_1'} B[y(\sigma_i), \sigma_i] - \sum_{\pi_2'} B[y(\sigma_i), \sigma_i] \\ &\subset \sum_{\pi_V} B[U_V, \sigma_i] + V + V \subset V + V + V \subset 4V,\end{aligned}$$

where the next to last inclusion follows by (1). Since V is arbitrary, this completes the proof.

If condition B3 is strengthened so that $B[y, \sigma]$ is completely additive *uniformly* for $y\in U$, where U is an arbitrary element of $\mathcal{U}$ (we say that $B[y, \sigma]$ is *uniformly completely additive*), then we can prove that boundedness[30] of $y(\sigma)$ implies B-summability and thus obtain the following theorem.

13.4. THEOREM. *Let $\mathfrak{X}$ be complete relative to $\mathcal{U}$ and let $B[y, \sigma]$ be uniformly*

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(29) It is easy to prove, using B1–B4, that $B[y, \sigma_0]=\theta$ for every $y\in\mathfrak{Y}$ implies $B[y, \sigma]=\theta$ for every $\sigma\subset\sigma_0$ and $y\in\mathfrak{Y}$.

(30) The function $y(\sigma)$ is said to be *bounded* provided the set $y(M)$ is bounded.

completely additive. Then, if $y(\sigma)$ is bounded, almost separable (B) and measurable $(\mathfrak{M})$ on the set $M\cap C\sigma_0$, it follows that $y(\sigma)$ is $\mathcal{V}_B$-integrable on each σ.

PART IV. RELATION TO OTHER INTEGRALS

14. **The Kolmogoroff integral.** The following theorem is a direct consequence of definitions; therefore the proof will be omitted.

14.1. THEOREM. *If $\mathfrak{X}$ is taken to be the real numbers, then the $\mathfrak{S}\mathcal{V}$-integral includes*[31] *the Kolmogoroff single-valued integral* [17, p. 663].

15. **The Phillips integral.** Consider the special "bilinear" functions $m(\sigma)x$, where $m(\sigma)$ is a completely additive positive measure function over $\mathfrak{M}$ and x is an element of the convex linear topological space $\mathfrak{X}$.

15.1. LEMMA. *Let X, $Y\subset\mathfrak{X}$, $\sigma=\bigcup_{i=1}^n\sigma_i$, $\sigma_i\cap\sigma_j=0$ $(i\neq j)$. Then $Y+m(\sigma)X\subset V_{cl}$, where $V\in\mathcal{V}$, implies $Y+\sum_{i=1}^n m(\sigma_i)X\subset V_{cl}$.*

We have $m(\sigma_i)Y+m(\sigma)m(\sigma_i)X\subset m(\sigma_i)V_{cl}$ $(i=1,\cdots,n)$. Summing these relations over i gives

$$\sum_{i=1}^n m(\sigma_i)Y+m(\sigma)\sum_{i=1}^n m(\sigma_i)X\subset\sum_{i=1}^n m(\sigma_i)V_{cl}.$$

But V_{cl} is convex; therefore $\sum_{i=1}^n m(\sigma_i)V_{cl}=m(\sigma)V_{cl}$. Moreover $m(\sigma)Y\subset\sum_{i=1}^n m(\sigma_i)Y$; hence $Y+\sum_{i=1}^n m(\sigma_i)X\subset V_{cl}$, provided $m(\sigma)\neq 0$. Since the lemma is obviously true if $m(\sigma)=0$, this completes the proof.

15.2. THEOREM. *The "bilinear" function $B[x,\sigma]=m(\sigma)x$ satisfies all of the conditions* B1–B5 (*including uniform complete additivity*); *therefore the entire theory of the $\mathcal{V}_B$-integral applies.*

All of the conditions are obviously satisfied except B4 which follows directly from Lemma 15.1. Observe that in the present case the constant of B4 can be taken as $1+e$, where $e>0$ is arbitrary.

Definition 10.1 of the $\mathcal{V}_B$-integral reduces in this case to precisely the definition used by Phillips [12, p. 118]; therefore

15.3. THEOREM. *For the case $B[x,\sigma]=m(\sigma)x$, the $\mathcal{V}_B$-integral reduces to the Phillips integral.*

16. **The Price integral.** Consider the special "bilinear" function $\tau(\sigma)x$, where x is an element of a Banach space $\mathfrak{X}$ with its norm topology of spheres having center θ and where $\tau(\sigma)$ is a linear continuous transformation of $\mathfrak{X}$ into itself. $\tau(\sigma)x$ means the result of transforming x by $\tau(\sigma)$. G. B. Price has

[31] One integral notion is said to include a second provided every function integrable according to the second notion is also integrable according to the first and to the same value.

defined an integral for this situation by first subjecting $\tau(\sigma)$ to the following conditions[32] [13, properties (8.1)–(8.3)].

T1. *If $\tau(\sigma)$ is the identically zero transformation, then $\sigma' \subseteq \sigma$ implies that $\tau(\sigma')$ is also identically zero.*

T2. *If $\tau(\sigma)$ is not the identically zero transformation, then it has a continuous inverse $\tau^{-1}(\sigma)$.*

T3. *For every sequence $\{\sigma_n\}$ of disjoint elements of $\mathfrak{M}$, $\tau(\bigcup \sigma_n) = \sum \tau(\sigma_n)$, where the series is unconditionally convergent according to the norm topology in the space of transformations.*

T4. *The generalized convex operator C^* generated by $\tau(\sigma)$ is bounded* [13, pp. 7–10]. *The bound will be denoted by β'.*

16.1. THEOREM. *If $\tau(\sigma)$ satisfies* T1–T4, *then the "bilinear" function $B[x, \sigma] = \tau(\sigma)x$ satisfies conditions* B1–B5 (*including uniform complete additivity*). *Therefore the entire theory of the $\mathcal{V}_B$-integral applies.*

All of the conditions are obviously satisfied except B4 for which we make the following proof.

Let $X_i \subset \mathfrak{X}$, $\sigma_i = \bigcup_{j=1}^{n_i} \sigma_i^j$, $\sigma_i \cap \sigma_j = 0$ $(i \neq j)$, $\sigma_i^j \cap \sigma_i^k = 0$ $(j \neq k)$, and assume $\sum_{i=1}^m \tau(\sigma_i) X_i \subset V_r$, where V_r is a sphere of radius r and center $\theta \in \mathfrak{X}$. The thing to be proved is that[33] $\sum_{i=1}^m \sum_{j=1}^{n_i} \tau(\sigma_i^j) X_i \subset \beta'(V_r)_{cl}$. In view of T1, we can evidently assume $\tau(\sigma_i)$ not identically zero $(i = 1, \cdots, m)$. For simplicity let $\mu_i^j = \tau(\sigma_i^j)\tau^{-1}(\sigma_i)$, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \tau(\sigma_i^j) X_i = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \mu_i^j \tau(\sigma_i) X_i.$$

Moreover, since $\sum_{j=1}^{n_i} \mu_i^j = I$, where I is the identity transformation, we have the following

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} \mu_i^j \tau(\sigma_i) X_i \subset \sum_{j_1=1}^{n_i} \cdots \sum_{j_m=1}^{n_m} \mu_1^{j_i} \cdots \mu_m^{j_m} \left(\sum_{i=1}^{m} \tau(\sigma_i) X_i \right)$$
$$\subset \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} \mu_1^{j_1} \cdots \mu_m^{j_m} V_r.$$

But $\{\mu_1^{j_1} \cdots \mu_m^{j_m}\}$ is obviously an element of the generalized convex operator C^*; therefore $\sum_{j=1}^m \sum_{j=1}^{n_i} \tau(\sigma_i^j) X_i \subset \beta'(V_r)_{cl}$.

The following theorem is an easy consequence of Theorems (11.4) and (4.11) of Price's paper.

(32) Condition T1 was not stated explicitly by Price but is implicit in the proof of part 6.9 of his Theorem 6.4. Also the situation discussed here is a bit more restricted than that considered by Price, since we require $\tau(\sigma)$ to be defined for every $\sigma \in \mathfrak{M}$ while Price admitted certain sets with "infinite measure" (see Footnote 14 above).

(33) Observe that the constant β of condition B4 can then be taken as $\beta' + e$, where $e > 0$ is arbitrary.

16.2. THEOREM. *If $B[x, \sigma] = \tau(\sigma)x$, then the $\mathcal{V}_B$-integral includes the Price integral.*

17. **Open questions.** Is it possible to dispense throughout with the condition that the space $\mathfrak{X}$ be convex?

Kolmogoroff [7[1]] has also given a definition of a multi-valued integral for real functions. What is the precise relationship of this Kolmogoroff integral to the $\mathcal{V}$-integral when $\mathfrak{X}$ is the real numbers?

Is the specialization of the $\mathcal{V}_B$-integral which includes the Price integral actually equivalent to it?

Is condition T4 on $\tau(\sigma)$ equivalent, in the presence of T1–T3, to condition B4 on $B[x, \sigma] = \tau(\sigma)x$?

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